A New Analytical Method for Self-Force Regularization. I

—— *Charged Scalar Particles in Schwarzschild Spacetime* ——

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We formulate a new analytical method for regularizing the self-force acting on a particle of small mass μ orbiting a black hole of mass M, where $\mu \ll M$. At first order in μ, the geometry is perturbed and the motion of the particle is affected by its self-force. The self-force, however, diverges at the location of the particle, and hence should be regularized. It is known that a properly regularized self-force is given by the tail part (or the R-part) of the self-field, obtained by subtracting the direct part (or the S-part) from the full self-field. The most successful method of regularization proposed so far relies on the spherical harmonic decomposition of the self-force, the so-called mode-sum regularization or mode decomposition regularization. However, except for some special orbits, no systematic analytical method for computing the regularized self-force has been constructed. In this paper, utilizing a new decomposition of the retarded Green function in the frequency domain, we formulate a systematic method for the computation of the self-force in the time domain. Our method relies on the post-Newtonian (PN) expansion, but the order of the expansion can be arbitrarily high. To demonstrate the essence of our method, in this paper, we focus on a scalar charged particle on the Schwarzschild background. Generalization to the gravitational case is straightforward, except for some subtle issues related with the choice of gauge (which exists irrespective of regularization methods).

§1. Introduction

We are now at the dawn of gravitational wave astronomy/astrophysics. The interferometric gravitational wave detectors LIGO,[1)] TAMA300[2)] and GEO600[3)] are currently in the early stage of their operations, and VIRGO[4)] is expected to be in operation soon. Furthermore, R&D studies of a space-based interferometer project,

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LISA,[5] are in rapid progress. It is expected that these interferometers, after their final sensitivity goals are achieved, will detect gravitational waves from compact star binaries and compact stars orbiting super-massive black holes.

To fully utilize the information contained in the observed gravitational wave data, and particularly for the purpose of precision testing general relativity, it is essential to have accurate theoretical predictions of the waveforms.[6] For nearly equal mass binaries, the standard PN approximation is a powerful tool to compute the waveforms.[7] An alternative method of computing the waveforms is the black hole perturbation approach.[8–13] This approach is very effective, in particular, when the mass ratio of the objects composing a binary is extreme. In this paper, considering the case of such extreme mass ratios, we propose a new method for calculating the corrections to the force acting on the small mass body (which is treated as a point particle) that are induced by the field generated by the particle itself, the so-called self-force corrections.

In the black hole perturbation approach, one can appeal to the energy-angular momentum balance argument to evaluate the radiation reaction to the orbit of a particle, namely, by equating the rates of change of the energy and angular momentum of the particle with those carried away by the gravitational waves emitted by it. However, the balance argument is not sufficient. First, the radiation reaction to the Carter constant can not be calculated with this method. The Carter constant is the third constant of motion of a test particle in the Kerr spacetime. The other two constants of motion, the energy and angular momentum with respect to the symmetry axis, are associated with the Killing vectors of the background spacetime, and their rates of change may be evaluated from the gravitational waves emitted to infinity or absorbed at the black hole horizon. In contrast, there is no correspondence to any such Killing vector in the case of the Carter constant. Hence, its rate of change is not directly related to the waves emitted. Second, and most importantly, the balance argument can yield only time-averaged rates of change of the two constants of motion, while there are many situations in which knowledge of the actual radiation reaction force, as well as the so-called conservative part of the self-force, at each instant of the orbital motion becomes necessary.

Consider a particle having either a scalar, electromagnetic or gravitational charge. The orbital motion of the particle will create a field at first order in its charge, and the motion will be affected by its self-field. This self-field, however, is divergent at the location of the particle. Hence, the force due to this self-field is apparently ill-defined. It is known that the self-force in the vicinity of the particle may be decomposed into the so-called direct part and the tail part, and that the correctly regularized self-force is given by the tail part. The justification of this prescription is given in Ref. 14) for the scalar and electromagnetic cases, and in Ref. 15) and 16) for the gravitational case.

In the scalar case, the Klein-Gordon equation is hyperbolic from the very beginning. In the electromagnetic and gravitational cases, the field equations can be put into hyperbolic form by choosing the Lorenz gauge (often called the 'harmonic gauge' in the gravitational case). In general, for a hyperbolic equation, the retarded Green function $G^{\mathrm{ret}}(x,x')$ can be split into two distinct pieces (at least locally when the two

points, namely, the field point x and the source point x' are sufficiently close), which is called the Hadamard form. One piece has support only on the future light-cone of x', and the other piece has support in the interior of the future light-cone of x'. The former gives the direct part and the latter the tail part. Recently, an equivalent but more elegant decomposition of the Green function was proposed,[17] in which the direct part is replaced by the S-part and the tail part by the R-part. The S-part is defined by adding a piece that has support outside the light-cone in such a way that it does not contribute to the self-force when it is subtracted from the full field. The remaining part is called the R-part, which now has support outside the light-cone as well. The advantage of this new decomposition is that the S-part is symmetric with respect to x and x', and it satisfies the same equation as the retarded Green function. This implies that the R-part now satisfies a source-free, homogeneous equation.

Thus our task is to evaluate the tail or R-part of the field and the self-force due to this. However, we do not have any systematic method to compute the R-part directly. In contrast, there exist several schemes to compute the full retarded field in Schwarzschild and Kerr spacetimes.[8)–13), 18)–21)] Therefore, what one can do is subtract the direct or S-part from the full field to compute the regularized self-force. There have been many investigation of this method.[22)–28)]

Because both the full field and the S-part diverge at the location of the particle, it is necessary to develop a regularization scheme to compute the difference between the two. The most successful scheme of regularization that has emerged is the mode decomposition (or mode-sum) regularization.[22), 23)] The full field can be decomposed into partial waves by using the spherical harmonics $Y_{\ell m}(\Omega)$. The contribution to the force from each ℓ-mode does not diverge in the coincidence limit of the field point with the location of the particle. Now, if we subtract the S-part from the full field, before summing over ℓ, the divergence disappears. Hence, we can perform summation over ℓ to obtain the expression for the regularized force.

The S-part can be calculated only in the form of a local expansion, that is, at field points sufficiently close to a point on the orbit, and therefore it is necessary to extend it over a sphere containing the orbital point to obtain the spherical harmonic coefficients. In recent years, a method to carry out its harmonic decomposition has been developed in the Schwarzschild case,[22)–27)] and it was recently extended to the case of the Kerr background.[28)] The full field is calculated using either the Regge-Wheeler-Zerilli or Teukolsky formalism, which uses the spherical (or spheroidal in the Kerr case) harmonic decomposition.

To this time, the regularized self-force has only been calculated numerically. These numerical results are important. However, an analytical understanding will be very useful. Analytical results obtained to this time are restricted to the case of particles orbiting in very special orbits (mainly circular or radial infall), and there are no results for the general case. This is primarily due to the mismatch in the schemes used for evaluating the full Green function and the S-part. The Regge-Wheeler-Zerilli and Teukolsky formalism rely heavily on the Fourier decomposition of the time-dependence by taking full advantage of the stationarity of the background spacetime, and therefore the full field is calculated in the frequency domain. Contrastingly, the S-part is in the time domain. If the orbit is specified a priori, it

is in principle possible to obtain the full field in the time domain by explicitly performing the integration (or summation) over the frequency ω. However, in practice, the explicit integration over ω is possible only in very special cases, such as circular orbits.[29),30)]

In this paper, we propose a new method to go from the frequency domain to time domain, and regularize analytically in order to calculate the self-force completely analytically. Though we employ the PN expansion method, it is formulated systematically in such a way that the order of the expansion can be taken arbitrarily high, as long as it is kept finite. In fact, our method is more effective in the far zone, and thus it supplements the previous numerical method, which is effective near the plunge. For simplicity, we restrict our analysis to the case of a geodesic orbit in the background spacetime, but removing this restriction is straightforward (although, of course, the equations become much more lengthy). Furthermore, we focus on the case of a scalar charge in order to avoid the gauge problem. There is a subtle problem associated with the choice of gauge in the gravitational case,[31),32)] but our method for integration over ω is equally applicable to the gravitational case, despite this problem.

The key idea of the new approach developed in this paper is to separate the retarded Green function in the frequency domain into two distinct pieces, in analogy to the S-R decomposition in the space-time domain. We call them the $\tilde{S}$-part and the $\tilde{R}$-part. The former contains all the singular terms to be subtracted, while the latter satisfies the source-free, homogeneous equation. In particular, once the PN order is specified, the contributions from only a finite number of ℓ is necessary to evaluate the $\tilde{R}$-part. The most important point of this new decomposition is that the $\tilde{S}$-part in the frequency domain is given in the form of a simple Taylor series with respect to ω multiplied by $\exp[-i\omega(t-t')]$. Therefore, the integration over ω can be performed easily for such terms. They just produce $\delta(t-t')$ and its derivatives. Using this technique, we can obtain the $\tilde{S}$-part in the time domain relatively easily. Then, the regularization is done by subtracting the S-part from the thus obtained $\tilde{S}$-part in the time domain.

This paper is organized as follows. In §2, we describe our new decomposition of the retarded Green function in the frequency domain. In §3, focusing on the scalar case, we demonstrate our regularization method. We first integrate the force due to the $\tilde{S}$-part over ω to obtain the harmonic modes of the force in the time domain, then subtract the S-part mode-by-mode, and finally sum over ℓ to obtain the force due to the $(\tilde{S}-S)$-part. We do not discuss the $\tilde{R}$-part, because it is finite from the beginning. The final section, §4, is devoted to conclusions and discussion. Some formulas and proofs of several propositions used in the text are given in appendices. For readers' convenience, formulas for 4PN order calculations can be found at the http://www2.yukawa.kyoto-u.ac.jp/~misao/BHPC/.

§2. New decomposition of the Green function in the frequency domain

We consider a point scalar charge q moving in the Schwarzschild background

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad (2\cdot1)$$

where $\{x^\alpha\} = \{t, r, \theta, \phi\}$ are the Schwarzschild coordinates, and M is the black hole mass. The full scalar field induced by this charged particle is given, using the retarded Green function, by

$$\psi^{\rm full}(x) = -q\int d\tau\, G^{\rm full}(x, z(\tau)), \qquad (2\cdot2)$$

where τ is the proper time of the particle, and $G^{\rm full}(x, x')$ satisfies the Klein-Gordon equation,

$$\nabla^\alpha\nabla_\alpha G^{\rm full}(x, x') = -\frac{\delta^{(4)}(x-x')}{\sqrt{-g}}, \qquad (2\cdot3)$$

with retarded boundary conditions. The full Green function is represented in terms of the Fourier-harmonic decomposition as

$$G^{\rm full}(x, x') = \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\sum_{\ell m} g^{\rm full}_{\ell m\omega}(r, r') Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta', \phi'). \qquad (2\cdot4)$$

Here, $Y_{\ell m}(\theta, \phi)$ are the ordinary spherical harmonics. Then, Eq. (2·3) reduces to an ordinary differential equation for the radial Green function as

$$\left[\left(1-\frac{2M}{r}\right)\frac{d^2}{dr^2} + \frac{2(r-M)}{r^2}\frac{d}{dr} + \left(\frac{\omega^2}{1-\dfrac{2M}{r}} - \frac{\ell(\ell+1)}{r^2}\right)\right] g^{\rm full}_{\ell m\omega}(r, r') = -\frac{1}{r^2}\delta(r-r'). \qquad (2\cdot5)$$

The radial part of the full Green function can be expressed in terms of homogeneous solutions of Eq.(2·5), which can be obtained using a systematic analytic method developed in Ref. 18). We have

$$\begin{aligned} g^{\rm full}_{\ell m\omega}(r, r') &= \frac{-1}{W_{\ell m\omega}(\phi^\nu_{\rm in}, \phi^\nu_{\rm up})}\left(\phi^\nu_{\rm in}(r)\phi^\nu_{\rm up}(r')\theta(r'-r) + \phi^\nu_{\rm up}(r)\phi^\nu_{\rm in}(r')\theta(r-r')\right), \\ W_{\ell m\omega}(\phi^\nu_{\rm in}, \phi^\nu_{\rm up}) &= r^2\left(1-\frac{2M}{r}\right)\left[\left(\frac{d}{dr}\phi^\nu_{\rm up}(r)\right)\phi^\nu_{\rm in}(r) - \left(\frac{d}{dr}\phi^\nu_{\rm in}(r)\right)\phi^\nu_{\rm up}(r)\right]. \end{aligned} \qquad (2\cdot6)$$

Here, the in-going and up-going homogeneous solutions are denoted, respectively, by $\phi^\nu_{\rm in}$ and $\phi^\nu_{\rm up}$, and ν is called the 'renormalized angular momentum',[18),19)] which is equal to ℓ in the limit $M\omega \to 0$.

We express the homogeneous solutions $\phi_{\rm in}^{\nu}$ and $\phi_{\rm up}^{\nu}$ in terms of the Coulomb wave functions $\phi_{\rm c}^{\nu}$ and $\phi_{\rm c}^{-\nu-1}$;[13),18)]

$$\begin{aligned}\phi_{\rm in}^{\nu} &= \alpha_\nu \phi_{\rm c}^{\nu} + \beta_\nu\, \phi_{\rm c}^{-\nu-1}\,,\\ \phi_{\rm up}^{\nu} &= \gamma_\nu\, \phi_{\rm c}^{\nu} + \delta_\nu \phi_{\rm c}^{-\nu-1}\,. \end{aligned} \tag{2·7}$$

The properties and the relations of the coefficients $\{\alpha_\nu,\, \beta_\nu,\, \gamma_\nu,\, \delta_\nu\}$ are studied extensively in Ref. 18) and 19). (The function $\phi_{\rm c}^{\nu}$ is denoted by R_c^{ν} in Ref. 18)) Here, we need to stress a remarkable property of the wave function ϕ_c^{ν}, which becomes manifest when we consider the PN expansion, i.e., when ϕ_c^{ν} is expanded in terms of $z := \omega r$ and $\epsilon := 2M\omega$, assuming they are small [$O(v)$ and $O(v^3)$, respectively]. In the expression of its PN expansion, $\Phi^{\nu} := (2z)^{-\nu}\phi_c^{\nu}$ contains only terms that are integer powers of z and ϵ, and there are no terms like $\log z$. In fact, we find that this condition, that $\log z$ is absent, uniquely specifies a single solution of the radial homogeneous equation. As is explained in Appendix A, this fact can be used to compute Φ^{ν}, simultaneously determining the eigenvalue ν. Furthermore, this PN expansion turns out to be a double Taylor series expansion in z^2 and ϵ/z, i.e., with only positive powers of ω^2. With the normalization $\Phi^{\nu} \to 1$, at the leading order, the expansion is

$$\begin{aligned}\Phi^{\nu} &= 1 - \frac{z^2}{2(2\ell+3)} - \frac{\ell\epsilon}{2z} + \frac{z^4}{8(2\ell+3)(2\ell+5)} + \frac{(\ell^2-5\ell-10)\epsilon z}{4(2\ell+3)(\ell+1)} + \cdots\,,\\ \nu &= \ell - \frac{15\ell^2+15\ell-11}{2(2\ell-1)(2\ell+1)(2\ell+3)}\epsilon^2 + \cdots\,. \end{aligned} \tag{2·8}$$

The solution $\phi_{\rm c}^{-\nu-1}$ can be obtained through the replacement $\ell \to -\ell-1$. The Wronskian of $\phi_{\rm c}^{\nu}$ and $\phi_{\rm c}^{-\nu-1}$ becomes

$$\begin{aligned}\omega\, W_{\ell m\omega}(\phi_{\rm c}^{\nu}, \phi_{\rm c}^{-\nu-1}) &= -\frac{2\ell+1}{2}\\ &+\frac{(496\ell^6+1488\ell^5+1336\ell^4+192\ell^3-757\ell^2-605\ell+338)\epsilon^2}{16(2\ell-1)^2(2\ell+1)(2\ell+3)^2} + \cdots\,. \end{aligned} \tag{2·9}$$

We note, however, that the general expression for the PN expansion of $\phi_{\rm c}^{-\nu-1}$ with this method (i.e., requiring the absence of $\log z$ terms) becomes invalid at the $(\ell-1)$th PN order. For small ℓ ($\leq$ PN + 1), the computation must be done following the systematic method given in Ref. 18). The respective results for the $\ell = 0$ and 1 cases are

$$\begin{aligned}\Phi^{\nu} &= \frac{7}{9} - \frac{7z^2}{54} - \frac{7\epsilon}{27z} + \frac{7z^4}{1080} - \frac{14\epsilon z}{27} + \cdots\,,\\ \Phi^{-\nu-1} &= -\frac{2z}{3\epsilon} j_0(z) + 1 + \frac{z^2}{18} + \frac{\epsilon}{2z} + \cdots\,,\\ \nu &= -\frac{7}{6}\epsilon^2 + \cdots\,, \quad \omega W = -\frac{49}{162} + \frac{23263}{29169}\epsilon^2 + \cdots\,, \end{aligned} \tag{2·10}$$

and

$$\Phi^{\nu} = 1 - \frac{z^2}{10} - \frac{\epsilon}{2z} + \frac{z^4}{280} - \frac{7\epsilon z}{20} + \cdots\,,$$

$$
\begin{aligned}
\Phi^{-\nu-1} &= -\frac{30z^3}{19\epsilon}j_1(z) + 1 + \frac{29z^2}{38} + \frac{\epsilon}{z} + \cdots\,, \\
\nu &= 1 - \frac{19}{30}\epsilon^2 + \cdots\,, \quad \omega W = -\frac{3}{2} + \frac{117443}{216600}\epsilon^2 + \cdots\,.
\end{aligned} \tag{2·11}
$$

Here, $j_0(z)$ and $j_1(z)$ are the spherical Bessel functions.

We now divide the Green function into two parts, as

$$
g^{\mathrm{full}}_{\ell m\omega}(r,r') = g^{\tilde{S}}_{\ell m\omega}(r,r') + g^{\tilde{R}}_{\ell m\omega}(r,r')\,, \tag{2·12}
$$

where

$$
\begin{aligned}
g^{\tilde{S}}_{\ell m\omega}(r,r') &= \frac{-1}{W_{\ell m\omega}(\phi^{\nu}_{\mathrm{c}},\phi^{-\nu-1}_{\mathrm{c}})} \\
&\quad \times \Big[\phi^{\nu}_{\mathrm{c}}(r)\phi^{-\nu-1}_{\mathrm{c}}(r')\theta(r'-r) + \phi^{-\nu-1}_{\mathrm{c}}(r)\phi^{\nu}_{\mathrm{c}}(r')\theta(r-r')\Big],
\end{aligned} \tag{2·13}
$$

$$
\begin{aligned}
g^{\tilde{R}}_{\ell m\omega}(r,r') &= \frac{-1}{(1-\tilde{\beta}_\nu\tilde{\gamma}_\nu)W_{\ell m\omega}(\phi^{\nu}_{\mathrm{c}},\phi^{-\nu-1}_{\mathrm{c}})}\Big[\tilde{\beta}_\nu\tilde{\gamma}_\nu\left(\phi^{\nu}_{\mathrm{c}}(r)\phi^{-\nu-1}_{\mathrm{c}}(r') + \phi^{-\nu-1}_{\mathrm{c}}(r)\phi^{\nu}_{\mathrm{c}}(r')\right) \\
&\quad + \tilde{\gamma}_\nu\phi^{\nu}_{\mathrm{c}}(r)\phi^{\nu}_{\mathrm{c}}(r') + \tilde{\beta}_\nu\phi^{-\nu-1}_{\mathrm{c}}(r)\phi^{-\nu-1}_{\mathrm{c}}(r')\Big]\,.
\end{aligned} \tag{2·14}
$$

Here, we have assumed that $\alpha_\nu \neq 0$ and $\delta_\nu \neq 0$, and we have also introduced the coefficients $\{\tilde{\beta}_\nu,\,\tilde{\gamma}_\nu\} := \{\beta_\nu/\alpha_\nu,\,\gamma_\nu/\delta_\nu\}$. Using results obtained in Ref. 18), we obtain the behavior of the coefficients $\{\tilde{\beta}_\nu,\,\tilde{\gamma}_\nu\}$ in the PN expansion as

$$
\tilde{\beta}_\nu = O(v^{6\ell+3})\,, \quad \tilde{\gamma}_\nu = O(v^{-3})\,. \tag{2·15}
$$

The functions ϕ^{ν}_{c} and $\phi^{-\nu-1}_{c}$ are, respectively, of $O(v^{\ell})$ and $O(v^{-\ell-1})$ (except at $\ell = 0$). Therefore, the three terms in the $\tilde{R}$-part of the Green function become, respectively, of $O(v^{6\ell})$, $O(v^{2\ell-2})$ and $O(v^{4\ell+2})$ relative to the $\tilde{S}$-part.

The part that we need to consider in the regularization of the force is just $\tilde{S}$. There is no divergence associated with the remaining $\tilde{R}$-part, and it terminates at finite ℓ as long as we restrict our consideration to finite PN order. Moreover, the $\tilde{R}$-part satisfies the homogeneous radial equation. This fact will be an additional advantage of the present method when we consider extension to the case of gravity. Because the $\tilde{R}$-part is a homogeneous solution, we can apply Chrzanowski's method[20)] to reconstruct the metric perturbations even in the frequency domain. We discuss this point in more detail in a separate paper.

§3. Computation of the $\tilde{S}$-part

We now compute the force due to the $\tilde{S}$-part for a general orbit. When expanded in terms of the spherical harmonics, the force corresponding to the S-part (the S-force) is known to take the form[23)]

$$
\lim_{x\to z_0} F^{S}_{\alpha,\ell} = A_\alpha L + B_\alpha + D_{\alpha,\ell}, \tag{3·1}
$$

where $F^{S}_{\alpha\ell}$ is the ℓ-mode of the S-force, $L = \ell + 1/2$, and A_α and B_α are independent of L. When summed over ℓ, the A-term gives rise to a quadratic divergence, and the

B-term diverges linearly. For large ℓ, $D_{\alpha\ell}$ is at most of $O(L^{-2})$, and hence it contains no divergence. Because the S-part can be calculated only locally, its extension to the entire sphere involves some ambiguity. As a result, the coefficient of each ℓ-mode, $D_{\alpha\ell}$, depends on the method of extension, but the final result after summation over ℓ, which is determined by the local behavior of the field near the source location, does not. It is known that

$$\sum_{\ell=0}^{\infty} D_{\alpha,\ell} = 0. \tag{3·2}$$

The difference between the S-force and the $\tilde{S}$-force should be finite, because the $\tilde{R}$-force is finite. Thus, in general, the $\tilde{S}$-force must take the same form as the S-force

$$\lim_{x\to z_0} F^{\tilde{S}}_{\alpha,\ell} = A_\alpha L + B_\alpha + \tilde{D}_{\alpha,\ell}\,. \tag{3·3}$$

Below we confirm explicitly that both A_α and B_α for the $\tilde{S}$-force coincide with those for the S-force. Therefore, the force due to the $\tilde{S}$-part minus the S-part, which is finite, is given by

$$F^{\tilde{S}-S}_\alpha = \sum_{\ell=0}^{\infty} \lim_{x\to z_0}\left(F^{\tilde{S}}_{\alpha,\ell} - F^{S}_{\alpha,\ell}\right) = \sum_{\ell=0}^{\infty} \tilde{D}_{\alpha,\ell}. \tag{3·4}$$

3.1. *Force in the time domain*

To obtain an expression for the $\tilde{S}$-force in the form of Eq. (3·3), it is necessary to perform the ω integration explicitly. Here, the key fact is that there appears no fractional power of ω in the $\tilde{S}$-part. This is because we have chosen ϕ_c^{ν} and $\phi_c^{-\nu-1}$ as the two independent basis functions. As noted above, except for the overall fractional powers z^{ν} and $z^{-\nu-1}$, they contain only the terms with positive integer powers of ω^2. When we consider a product of these two functions, ω contained in the overall factors z^{ν} and $z^{-\nu-1}$ just produces ω^{-1}, which is canceled by ω from the inverse of the Wronskian. Thus, $g^{\tilde{S}}_{\ell m\omega}(r,r')$ is expanded as

$$g^{\tilde{S}}_{\ell m\omega}(r,r') = \sum_{k=0}^{\infty} \omega^{2k} \mathcal{G}_{\ell mk}(r,r'). \tag{3·5}$$

Therefore, the integration over ω can be performed easily, as we shall show now explicitly. The Fourier transform of ω^{2n} simply produces

$$\int d\omega\, \omega^{2n} e^{-i\omega(t-t')} = 2\pi(-1)^n \partial_{t'}^{2n} \delta(t-t').$$

Differentiation of the delta function in the expression above can be integrated by parts to act on the source term. Thus, we can express the $\tilde{S}$-force in the time domain as

$$F^{\tilde{S}}_{\alpha,\ell} = q^2 P_\alpha{}^\beta \lim_{x\to z(t)} \nabla_\beta \sum_{m,k} (-1)^k (\partial_t)^{2k} \frac{d\tau(t)}{dt} \mathcal{G}_{\ell mk}(r, z^r(t)) Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(z^\theta(t), z^\varphi(t)). \tag{3·6}$$

Here we have inserted the projection tensor $P_\alpha{}^\beta = \delta_\alpha{}^\beta + u_\alpha u^\beta$, where u^α is the four-velocity, so that the normalization $u^\alpha u_\alpha = -1$ is maintained. Now that we have the $\tilde{S}$-force given in the time domain, the meaning of the coincidence limit $r \to z^r(t)$ is transparent.

Here one can set $z^\theta(t) \equiv \pi/2$ without loss of generality. Then, nothing that $\partial_\varphi Y_{\ell m}(\pi/2,\varphi) = imY_{\ell m}(\pi/2,\varphi)$ and $\partial_\varphi Y^*_{\ell m}(\pi/2,\varphi) = -imY^*_{\ell m}(\pi/2,\varphi)$, the summation over m can be done by using the formulas

$$\sum_{m=-\ell}^{\ell} m^j \left| Y_{\ell m}\left(\frac{\pi}{2},\varphi\right)\right|^2 = \begin{cases} \lambda_{(j/2)}(\ell), & \text{for } j = \text{even}, \\ 0, & \text{for } j = \text{odd}, \end{cases}$$
$$\sum_{m=-\ell}^{\ell} m^j\, \partial_\theta Y_{\ell m}(\theta,\varphi)\Big|_{\theta=\pi/2} Y^*_{\ell m}\left(\frac{\pi}{2},\varphi\right) = 0, \tag{3·7}$$

where $\lambda_{(n)}(\ell)$ is a polynomial function of ℓ of order $n+1$ defined by

$$\sum_{n=0}^{\infty} \frac{\lambda_n z^{2n}}{n!} = \frac{2\ell+1}{4\pi} e^{\ell z}\, {}_2F_1\left(\frac{1}{2}, -\ell; 1; 1-e^{-2z}\right). \tag{3·8}$$

3.2. *Separation of the A-term*

Before performing the operation discussed in the preceding subsection, we separate the A-term from the other contributions. This can be done easily by using the fact that only the A-term has a jump in its values at the coincidence limit, depending on the direction from which the source point is approached. We divide the $\tilde{S}$-part Green function into symmetric and anti-symmetric parts as

$$g^{\tilde{S}}_{\ell m\omega}(r,r') = g^{\tilde{S}(+)}_{\ell m\omega}(r,r') + \mathrm{sgn}(r-r') g^{\tilde{S}(-)}_{\ell m\omega}(r,r'), \tag{3·9}$$

where $\mathrm{sgn}(y) = \pm 1$ for $y \gtrless 0$, and

$$g^{\tilde{S}(\pm)}_{\ell m\omega}(r,r') = \frac{-1}{2W_{\ell m\omega}(\phi^\nu_{\rm c}, \phi^{-\nu-1}_{\rm c})}\left[\phi^\nu_{\rm c}(r)\phi^{-\nu-1}_{\rm c}(r') \pm \phi^{-\nu-1}_{\rm c}(r)\phi^\nu_{\rm c}(r')\right]. \tag{3·10}$$

Then the force due to the anti-symmetric part must coincide exactly with the A-term. We know that the A-term for the S-part has a simple form proportional to L, while the expression for the $\tilde{S}$-force given in terms of the Green function is more complicated. The reason why such a simple result for the A-term is recovered is explained in Appendix B. We concentrate now on the symmetric part, which is responsible for the B and D-terms.

3.3. *The $(\tilde{S} - S)$-part of the force*

The result for the $\tilde{S}(+)$-part of the force is

$$F^{\tilde{S}(+)}_{t,\ell} = \frac{q^2 u^r}{4\pi r_0^2}\sum_{n=0}^{\infty} K_t^{(n)}(\ell), \quad F^{\tilde{S}(+)}_{\theta,\ell} = 0, \quad F^{\tilde{S}(+)}_{\varphi,\ell} = \frac{q^2 u^r \mathcal{L}}{4\pi r_0^2}\sum_{n=0}^{\infty} K_\varphi^{(n)}(\ell), \tag{3·11}$$

and

$$F^{\tilde{S}(+)}_{r,\ell} = -\frac{\mathcal{E}}{u^r(1-2M/r_0)} F^{\tilde{S}(+)}_{t,\ell} - \frac{\mathcal{L}}{u^r r_0^2} F^{\tilde{S}(+)}_{\varphi,\ell}, \tag{3·12}$$

where the coefficients $K^{(i)}_{\alpha,\ell}$, whose upper index (i) represents the PN order, are formally given by

$$K^{(n)}_{t,\ell} = \sum_{i+j+k=n} d_t^{(ijk)}(\ell)\ (\delta_{\mathcal{E}})^i \left(\frac{\mathcal{L}^2}{r_0^2}\right)^j U^k,$$
$$K^{(n)}_{\varphi,\ell} = \sum_{i+j+k=n} d_\varphi^{(ijk)}(\ell)\ (\delta_{\mathcal{E}})^i \left(\frac{\mathcal{L}^2}{r_0^2}\right)^j U^k. \tag{3·13}$$

Here, the quantities d_α are some functions of ℓ, $r_0 \equiv z^r(t_0)$, $\delta_{\mathcal{E}} \equiv 1-\frac{1}{\mathcal{E}^2}$, $U \equiv \frac{M}{r_0}$, and $\mathcal{E}$ and $\mathcal{L}$ are, respectively, the energy and the angular momentum of the particle. To obtain these expressions, we have used the first integrals of the geodesic equations,

$$\left(\frac{dz^r(t)}{dt}\right)^2 = \left(1-\frac{2M}{z^r(t)}\right)^2 - \frac{1}{\mathcal{E}^2}\left(1-\frac{2M}{z^r(t)}\right)^3\left(1+\frac{\mathcal{L}^2}{z^r(t)^2}\right),$$
$$\frac{dz^\varphi(t)}{dt} = \frac{\mathcal{L}}{\mathcal{E}}\frac{1}{z^r(t)^2}\left(1-\frac{2M}{z^r(t)}\right), \quad \frac{dt}{d\tau} = \frac{\mathcal{E}}{1-2M/z^r(t)}, \tag{3·14}$$

and we have also reduced higher-order derivatives with respect to t by using the equations of motion

$$\frac{d^2z^r(t)}{dt^2} = \frac{2M}{z^r(t)^2}\left(1-\frac{2M}{z^r(t)}\right) + \frac{\mathcal{L}^2}{\mathcal{E}^2 z^r(t)^3}\left(1-\frac{2M}{z^r(t)}\right)^3 - \frac{3M}{\mathcal{E}^2 z^r(t)^2}\left(1-\frac{2M}{z^r(t)}\right)^2\left(1+\frac{\mathcal{L}^2}{z^r(t)^2}\right). \tag{3·15}$$

As long as we ignore corrections that are higher order in μ, we can assume that the orbit is momentarily geodesic. Here, we find that the $\tilde{S}$-part force is written solely in terms of the orbit at the location of the particle (that is, no tails!). Hence, for the correction at lowest order in μ, the force coming from the $(\tilde{S}-S)$-part is also written in terms of the position and the velocity of the particle. We do not have any terms with a positive power of ℓ, as expected. However, this cancellation looks rather miraculous in the present formulation. In Appendix C, we give a brief explanation of why the terms with positive powers of ℓ are absent. From the asymptotic behavior for large ℓ, we can read off the coefficients B_α. These coefficients are identical to the results obtained previously.[22),25)] As mentioned earlier, a separate treatment is necessary for small ℓ. To summarize, the $(\tilde{S}-S)$-part of the force is given by

$$F^{\tilde{S}-S}_\alpha = \sum_{\ell=0}^{\infty}\lim_{x\to z_0}\left(F^{\tilde{S}}_{\alpha,\ell} - F^{S}_{\alpha,\ell}\right) = \sum_{\ell=0}^{\infty}\left(F^{\tilde{S}}_{\alpha,\ell} - A_\alpha\left(\ell+\frac{1}{2}\right) - B_\alpha\right)$$
$$= \sum_{\ell=0}^{\infty}\left(F^{\tilde{S}(+)}_{\alpha,\ell} - B_\alpha\right) \tag{3·16}$$

The summation over ℓ is performed by using the decomposition into partial fractions and the formulas

$$\sum_{\ell=1}^{\infty}\frac{1}{\ell^{2n}} = 2^{2n-1}\pi^{2n}\frac{B_n}{(2n)!}, \quad \sum_{\ell=1}^{\infty}\frac{1}{(\ell-1/2)^{2n}} = 2^{2n-1}(2^{2n}-1)\pi^{2n}\frac{B_n}{(2n)!}, \tag{3·17}$$

where n is a positive integer, and B_n is the Bernoulli number defined by

$$\frac{x}{e^x-1}+\frac{x}{2}-1=\sum_{n=1}^{\infty}2^{2n}(2^{2n}-1)\frac{B_n}{(2n)!}x^{2n}. \tag{3·18}$$

The reason why odd powers of ℓ do not arise is as follows. If we have a term like

$$\sum_{n=0}^{\infty}\frac{1}{(\ell+k/2)^n},$$

then, due to the symmetry under $\ell\to-(\ell+1)$, we also have a term

$$\sum_{n=0}^{\infty}\frac{1}{(-\ell-1+k/2)^n},$$

where k is an integer. If n is odd, these two contributions combine to leave a summation of finite terms. Hence, there remains no infinite summation of odd power terms in the final expression.

The result for the $(\tilde{S}-S)$-part of the scalar self-force is

$$F_t^{\tilde{S}-S}=\frac{q^2u^r}{4\pi r_0^2}\sum_{n=0}^{\infty}C_t^{\tilde{S}-S(n)},\quad F_\theta^{\tilde{S}-S}=0,\quad F_\varphi^{\tilde{S}-S}=\frac{q^2u^r\mathcal{L}}{4\pi r_0^2}\sum_{n=0}^{\infty}C_\varphi^{\tilde{S}-S(n)}, \tag{3·19}$$

and

$$F_r^{\tilde{S}-S}=-\frac{\mathcal{E}}{u^r(1-2M/r_0)}F_t^{\tilde{S}-S}-\frac{\mathcal{L}}{u^r r_0^2}F_\varphi^{\tilde{S}-S}, \tag{3·20}$$

where the coefficients $C_\alpha^{\tilde{S}-S(i)}$, whose upper index, i, represents the PN order, are formally given by

$$\begin{aligned}C_t^{\tilde{S}-S(n)}&=\sum_{i+j+k=n}e_t^{(ijk)}(\delta_{\mathcal{E}})^i\left(\frac{\mathcal{L}^2}{r_0^2}\right)^j U^k,\\ C_\varphi^{\tilde{S}-S(n)}&=\sum_{i+j+k=n}e_\varphi^{(ijk)}(\delta_{\mathcal{E}})^i\left(\frac{\mathcal{L}^2}{r_0^2}\right)^j U^k.\end{aligned} \tag{3·21}$$

Here e_α are some constant numbers.

Once we obtain the general expression for the $(\tilde{S}-S)$-part of the force, computation of the remaining $\tilde{R}$-part is rather easy, because only terms up to a finite value of ℓ contribute to the force for a given PN order. Then, the R-part of the force, which is what we want in the end, is given by

$$F_\alpha^R=F_\alpha^{\tilde{S}-S}+F_\alpha^{\tilde{R}}. \tag{3·22}$$

Now we discuss a technical but important property of the $\tilde{S}$-part of the Green function. The two independent radial functions at leading order in ϵ are given by the

spherical Bessel functions $j_\ell(z)$ and $n_\ell(z)$. Both ϕ_c^ν and $\phi_c^{-\nu-1}$ are given by linear combinations of these two independent solutions. Up to $O(\epsilon^0)$, we have $\phi_c^\nu \propto j_\ell(z)$, while $\phi_c^{-\nu-1} \propto \epsilon^{-1} j_\ell(z) + C_\ell\, n_\ell(z)$, where C_ℓ is a constant of order unity. Here, in passing, we note that the leading term of $\phi_c^{-\nu-1}$ in the PN expansion comes not from the term $\epsilon^{-1} j_\ell(z)$ but from $n_\ell(z)$ for $\ell \geq 2$. This is because the ratio of the two terms is $j_\ell(z)/(\epsilon n_\ell(z)) \propto z^{2\ell-2} = O(v^{2\ell-2})$ in the PN expansion. At first glance, the existence of the term $\epsilon^{-1} j_\ell(z)$ seems problematic, since it would naively lead to a term in the Green function that behaves as $1/M$. Collecting the leading terms in ϵ, we find that the Green function has a term proportional to $1/M$ of the form

$$\propto \frac{1}{M} \sum_{\ell m} \int d\omega e^{-i\omega(t-t')} j_\ell(\omega r) j_\ell(\omega r') Y_{\ell m}(\Omega) Y^*_{\ell m}(\Omega').$$

This expression is identical the radiative Green function in Minkowski space, except for an additional multiplicative factor $1/(M\omega)$. Therefore, after summation over ℓ and m and integration over ω, we find that this part is

$$\begin{aligned} \propto &\int dt' \left[\frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} - \frac{\delta(t-t'+|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \right] \\ &= \frac{\theta(t-t'+|\mathbf{x}-\mathbf{x}'|) - \theta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}. \end{aligned}$$

At lowest order in M, the trajectory of a particle is a straight line in the Minkowski background. Because the above expression for the leading part of the Green function is Lorentz invariant, we can choose this straight line to be static, without loss of generality. Then, it is easy to see that the component of the field proportional to $1/M$ is constant. Hence, this part does not contribute to the force.

Another important property of the force due to the $(\tilde{S} - S)$-part is that it contains only the conservative part of the force. To show this, we use the fact that the equations of motion take the form $\dot{u}^r =$ (a function even in u^r) and $\dot{\mathcal{E}} = \dot{\mathcal{L}} = 0$, at leading order in μ [see Eqs. (3·14) and (3·15)]. Here, the dot represents differentiation with respect to t. Recalling the general expression for the $\tilde{S}$-force in Eq. (3·6), we see that the t-component contains an odd number of time derivatives, $\partial^{2k+1}/\partial t^{2k+1}$, the r-component an even number of time derivatives plus one radial derivative, $\partial^{2k+1}/(\partial t^{2k}\partial r)$, and the φ-component an even number of time derivatives plus one φ derivative, $\partial^{2k+1}/(\partial t^{2k}\partial\varphi)$. Now, using the equations of motion, the t derivative can be replaced by

$$\begin{aligned} \frac{\partial}{\partial t} &= \dot{z}^r(t)\frac{\partial}{\partial z^r} + \ddot{z}^r(t)\frac{\partial}{\partial \dot{z}^r} + \dot{z}^\varphi(t)\frac{\partial}{\partial z^\varphi} \\ &= \dot{z}^r(t)\frac{\partial}{\partial z^r} + \ddot{z}^r(t)\frac{\partial}{\partial \dot{z}^r} - im\,\dot{z}^\varphi(t)\,, \end{aligned}$$

and the φ derivative by

$$\frac{\partial}{\partial\varphi} = +im\,.$$

Notice that one differentiation with respect to t or φ changes the total power of u^r and m by an odd number, while a differentiation with respect to r does not. Notice also that only the terms even in m remain after summation over m. Therefore the ℓ-mode of the $\tilde{S}$-force takes the form

$$F^{\tilde{S}}_{t\,\ell} = \mathcal{F}_{t\,\ell}(r,\mathcal{E},\mathcal{L})\,u^r\,, \quad F^{\tilde{S}}_{r\,\ell} = \mathcal{F}_{r\,\ell}(r,\mathcal{E},\mathcal{L}), \quad F^{\tilde{S}}_{\varphi\,\ell} = \mathcal{F}_{\varphi\,\ell}(r,\mathcal{E},\mathcal{L})\,u^r\,. \tag{3·23}$$

The S-part of the force is known to have exactly the same form. This implies that the $(\tilde{S}-S)$-part of the force also takes the same form. Thus, after summing over ℓ, we conclude that the final form of the $(\tilde{S}-S)$-part of the force is

$$F^{\tilde{S}-S}_{t} = \mathcal{F}_{t}(r,\mathcal{E},\mathcal{L})\,u^r\,, \quad F^{\tilde{S}-S}_{r} = \mathcal{F}_{r}(r,\mathcal{E},\mathcal{L}), \quad F^{\tilde{S}-S}_{\varphi} = \mathcal{F}_{\varphi}(r,\mathcal{E},\mathcal{L})\,u^r\,. \tag{3·24}$$

We can now explicitly show that the above form of the force implies the absence of a dissipative reaction effect. In other words, the force is conservative. The equations of motion to $O(\mu^2)$ are given by

$$\mu\frac{D}{d\tau}\tilde{u}^\mu = F^\mu, \tag{3·25}$$

where $\tilde{u}^\mu$ is the perturbed four velocity and $D/d\tau$ is the covariant derivative. Then, we obtain the evolution equation for the perturbed energy $\tilde{\mathcal{E}} := -\mu\,\hat{t}_\mu\tilde{u}^\mu$ as

$$\frac{d\tilde{\mathcal{E}}}{d\tau} = -\mu\frac{D}{d\tau}(\hat{t}_\mu\tilde{u}^\mu) = -\hat{t}_\mu F^\mu = -\mathcal{F}_t(r)\frac{dr}{d\tau}, \tag{3·26}$$

where $\hat{t}^\mu = (\partial_t)^\mu$ is the time-like Killing vector. This equation is integrated to give

$$\tilde{\mathcal{E}} = \mathcal{E} - \int^r \mathcal{F}_t(r)dr. \tag{3·27}$$

Here, $\mathcal{E}$ is an integration constant, which we can interpret as the unperturbed energy. In the same manner, for the perturbed angular momentum $\mathcal{L}$, we obtain

$$\tilde{\mathcal{L}} = \mathcal{L} + \int^r \mathcal{F}_\varphi(r)dr. \tag{3·28}$$

Thus we find that there is no cumulative effect on the evolution of the energy and angular momentum of the particle. In other words, a force of the form (3·24) preserves the presence of the constants of motion $\mathcal{E}$ and $\mathcal{L}$. Concerning the radial motion, $\tilde{u}^r$ can be expressed in terms of $\mu\,\tilde{u}_t = \tilde{\mathcal{E}}$ and $\mu\,\tilde{u}_\varphi = \tilde{\mathcal{L}}$ by using the normalization condition of the four velocity. We have

$$\mu\,\tilde{u}^r = \pm\left[\tilde{\mathcal{E}}^2 - (1-2M/r)\left(1+\tilde{\mathcal{L}}^2/r^2\right)\right]^{1/2}. \tag{3·29}$$

Thus, $\tilde{u}^r$ is obtained as a function of r.

§4. Conclusions and discussion

The present work is an attempt to make progress toward a more *realistic* (calculation effective) analytic scheme for constructing orbits, taking into account radiation reaction effects. The key idea proposed in this paper is a new decomposition of the Green function into $\tilde{S}$ and $\tilde{R}$-parts, given in Eq. (2·12). This new decomposition relies on a systematic analytic approach to the black hole perturbation developed in Refs. 13), 18) and 19). The new decomposition is not identical to the usual S and R decomposition,[17] but they share certain properties. The $\tilde{S}$-part is singular and symmetric, and it satisfies the same inhomogeneous equation as the Green function. The $\tilde{R}$-part is regular, and it satisfies the source-free equation. Considering a scalar charged particle, we showed that the $\tilde{S}$-part of the self-force can be evaluated analytically in the time domain and that it yields the same regularization parameters A_α and B_α in the mode-decomposition regularization as the usual S-part. This implies that the $\tilde{S}$-part contains all the singular behavior of the original S-part. Also, we showed that the self-force due to the $(\tilde{S} - S)$-part is conservative. Moreover, we found that the $\tilde{R}$-part of the force valid up to the $(\ell + 0.5)$th Post-Newtonian (PN) order can be obtained by taking account of only the spherical harmonic modes up to ℓ-th order.

The analysis here is restricted to the self-force due to a scalar field for its simplicity. The extension to the electromagnetic case is straightforward, although the computation of the force from the master variable of the perturbations[10] becomes more tedious. The gravitational case, however, is more complicated, because in that case, the self-force is gauge dependent. In the scalar case, as is manifest from our calculation, the computation of the S-part is in fact unnecessary if we make use of the fact that the non-singular part of the regularization parameter $D_{\alpha\ell}$ for the S-part vanishes after summing over ℓ modes. By virtue of this fact, the simple prescription of subtracting A- and B-terms is valid in this case. In the case of gravity, before subtracting the S-part from the $\tilde{S}$-part of the force, we need to adjust the gauges that are originally different. A natural prescription for this is to transform the S-part, originally given in Lorenz gauge, to the gauge in which the $\tilde{S}$-part is computed. Due to this gauge transformation, the parameter $D_{\alpha\ell}$ as well as the other regularization parameters will be altered. Then the contribution to the self-force from $D_{\alpha\ell}$ will not vanish in general. This needs to be studied in more detail.

Related to the gauge dependence of the gravitational self-force, there arises a conceptual question: What kind of gauge-independent concepts are contained in this otherwise gauge-dependent quantity? There are several constants of motion for geodesics in the background black hole spacetime. These constants of motion evolve after we incorporate the self-force. However, the secular change of "the constants of motion" has a gauge-invariant meaning. It can be evaluated by merely taking account of the force due to the $\tilde{R}$-part, or we can simply use the radiative Green function.[33] In addition to the information regarding the constants of motion, the self-force may contain other gauge-invariant information. This question needs to be answered, although it is not a problem specific to our present method. We may find that the physical information contained in the gauge-dependent self-force by itself is

very limited. Even if this is the case, calculation of the self-force will be a necessary step to develop a black hole perturbation theory to second order in the mass of the orbiting particle.

One of the main advantages of our method is that it allows a successful implementation of a systematic post-Newtonian expansion technique in the black hole perturbation. There could be criticism of our approach in regard to the limitation of the PN expansion itself. The black hole perturbation is considered to be a method complimentary to the standard post-Newtonian approximation. In this sense, one might think that there is no point in using the PN expansion in the black hole perturbation approach. However, we should stress that the PN expansion of the perturbation in the black hole spacetime can be systematically extended to an arbitrarily high order without any conceptual difficulties. Hence, if we can make use of this advantage, problems far beyond the validity of the finite order standard post-Newtonian approximation can be investigated.

In an actual computation, the achievable PN order will be limited. Then, the question will be the speed of the convergence of the PN expansion. In this regard, there are a couple of encouraging pieces of evidence. First, we mention the issue of the innermost stable circular orbit (ISCO). At present, the calculation of the orbital frequency at ISCO up to 3PN order has been made.[34)] The result is in rather good agreement with the numerical result for an equal-mass binary.[35)] Second, we mention the energy loss rate of a particle in a circular orbit in the Schwarzschild black hole evaluated from the asymptotic waveform.[12)] Although the convergence of the PN expansion becomes slower and slower for a smaller orbital radius, there is no evidence of the failure converge, even at ISCO.[36)] Hence, if we can develop a systematic method of evaluating the gravitational self-force in the PN expansion, its range of validity should be very wide.

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Appendix A
— An Easy Way to Find ϕ^ν for Sufficiently Large ℓ —

Consider a radial function $\phi^\nu(z)$ which satisfies

$$\left[\left(1-\frac{2M}{r}\right)\frac{d^2}{dr^2}+\frac{2(r-M)}{r^2}\frac{d}{dr}+\left(\frac{\omega^2}{1-\dfrac{2M}{r}}-\frac{\ell(\ell+1)}{r^2}\right)\right]\phi^\nu(r)=0\,. \quad \text{(A·1)}$$

As long as we consider sufficiently large ℓ, this radial wave function can be completely specified up to an overall normalization by the requirement that $\Phi^\nu := (2z)^{-\nu}\phi_c^\nu$ does not contain $\log z$ in its PN power series expansion with respect to z^2 and ϵ/z. The same condition simultaneously determines the renormalized angular momentum $\nu \approx \ell$. In fact, the equation for Φ^ν becomes

$$\begin{aligned}\left[z^2\partial_z^2+2z(\nu+1)\partial_z\right]\Phi^\nu = &\left[\frac{\epsilon}{z}(2-\frac{\epsilon}{z})z^2\partial_z^2+\frac{\epsilon}{z}\left\{(4\nu+3)-\frac{\epsilon}{z}(2\nu+1)\right\}z\partial_z\right.\\ &\left.+(\ell-\nu)(\ell+\nu+1)\left(1-\frac{\epsilon}{z}\right)-z^2+\frac{\epsilon}{z}\left(1-\frac{\epsilon}{z}\right)\nu^2\right]\Phi^\nu. \end{aligned}\quad \text{(A·2)}$$

The right-hand side of this equation is of higher order in the PN expansion. Substituting the Taylor expansion of Φ^ν with respect to ϵ/z and z^2 into the above equation, the coefficients are determined order by order. However, the z-independent terms in Φ^ν vanish on the left-hand side. Therefore, the terms which become zeroth order in z should also vanish on the right-hand side. This condition determines ν order by order. If, however, ℓ is not sufficiently large, terms proportional to $z^{-2\ell-1}$ arise. For such terms, the left-hand side is suppressed by a factor of $O(\epsilon^2)$. Hence, the iteration scheme becomes invalid for small ℓ, and we need to go back to the original method presented in Ref. 18).

Appendix B
— A-Term —

Here, we consider the A-term extracted from the $\tilde{S}$-part. As we have explained in the main text, the A-term corresponds to a jump of the field. Therefore, it is given by the antisymmetrized Green function

$$g_{\ell m\omega}^{\tilde{S}(-)}(r,r')=\frac{-1}{2W_{\ell m\omega}(\phi_c^\nu,\phi_c^{-\nu-1})}\left[\phi_c^\nu(r)\phi_c^{-\nu-1}(r')-\phi_c^{-\nu-1}(r)\phi_c^\nu(r')\right]\,. \quad \text{(B·1)}$$

We introduce a new function, $\chi_c := r\phi_c$, which satisfies

$$\left[\partial_{r^*}^2+\omega^2-V(r)\right]\chi_c(r^*)=0\,, \quad \text{(B·2)}$$

$$V(r)=\left(1-\frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}+\frac{2(r-M)}{r^3}\right)\,, \quad \text{(B·3)}$$

where $r^* = r + 2M \ln(r/2M - 1)$. The Wronskian is written in terms of r^* and χ_c as

$$W_{\ell m\omega}(\phi_c^\nu, \phi_c^{-\nu-1}) = \left(\frac{d}{dr^*}\chi_c^{-\nu-1}(r^*)\right)\chi_c^\nu(r^*) - \left(\frac{d}{dr^*}\chi_c^\nu(r^*)\right)\chi_c^{-\nu-1}(r^*)\,. \quad \text{(B·4)}$$

We expand the function $g_{\ell m\omega}^{\tilde{S}(-)}(r,r')$ in a power series with respect to $r^* - r^{*\prime}$ as

$$g_{\ell m\omega}^{\tilde{S}(-)}(r,r') = \sum_{n\geq 0} \boldsymbol{g}_n(r')(r^* - r^{*\prime})^n\,, \quad \text{(B·5)}$$

$$\boldsymbol{g}_n(r') = \frac{1}{n!}\partial_{r^*}^n g_{\ell m\omega}^{\tilde{S}(-)}(r,r')\bigg|_{r=r'}\,. \quad \text{(B·6)}$$

The higher-order derivatives with respect to ∂_{r^*} in $\boldsymbol{g}_n(r')$ can be reduced by using Eq. (B·2). Hence, either one or zero r^*-derivatives remain in the end. Setting $r = r'$, the terms with no r^* derivative vanish, while the terms with a single derivative yield the Wronskian, which cancels the denominator. As a result, we obtain a rather simple expression for $\boldsymbol{g}_n$. In fact, we have

$$\begin{aligned}
\boldsymbol{g}_1 &= -\frac{1}{2r'^2}\,,\\
\boldsymbol{g}_2 &= \left(1-\frac{2M}{r'}\right)\frac{1}{4r'^3}\,,\\
\boldsymbol{g}_3 &= -\frac{1}{12r'^2}\left[(-\omega^2+V(r')) - 2\left(1-\frac{2M}{r'}\right)\frac{r'-3M}{r'^3}\right]\,,\\
&\cdots.
\end{aligned}$$

Note that only even positive integer powers of ω appear. Therefore, let us consider terms proportional to ω^{2N} for a given N. Because the factor ω^{2N} arises only from the elimination of the $2N$ derivatives, and because only a single derivative can remain at the end of the calculation, ω^{2N} can be contained only in $\boldsymbol{g}_n(r')$ with $n \geq 2N+1$. Conversely, this means that we have

$$\begin{aligned}
\boldsymbol{g}_{2N}(r')\,(r^* - r^{*\prime})^{2N} &= \sum_{n=0}^{N-1}\omega^{2n}\,a_n(r')(r^* - r^{*\prime})^{2N}\,,\\
\boldsymbol{g}_{2N+1}(r')\,(r^* - r^{*\prime})^{2N+1} &= \sum_{n=0}^{N}\omega^{2n}\,b_n(r')(r^* - r^{*\prime})^{2N+1}\,,
\end{aligned} \quad \text{(B·7)}$$

where $a_n(r')$ and $b_n(r')$ are independent of ω. We replace ω with a time derivative, which acts on $r' = z^r(t)$. The force is calculated by differentiating the potential once. Hence, the term proportional to $\boldsymbol{g}_{2N}$ vanishes in the coincidence limit, because it contains $2N-1$ derivatives at most. As for the the term $\boldsymbol{g}_{2N+1}$, which contains $2N+1$ derivatives, each derivative must act on each factor of $r^* - r^{*\prime}$ in $(r^* - r^{*\prime})^{2N+1}$ to give a finite result. Therefore, in $\boldsymbol{g}_{2N+1}(r')$, we only need to keep b_N in the coincidence limit. Thus, we can simplify the coefficients $\boldsymbol{g}_n$ as

$$\boldsymbol{g}_{2N} \approx 0, \quad \boldsymbol{g}_{2N+1} \approx -\frac{1}{2r'^2(2N+1)!}(-\omega^2)^N. \quad \text{(B·8)}$$

Substituting this expression into Eq. (3·6), and summing over the m-modes, the anti-symmetric part of the force can be calculated as

$$F_{\alpha\ell}^{\tilde{S}(-)} = -\frac{q^2(2\ell+1)}{4\pi}\left[\partial_\alpha(r-z^r(t))\right]\sum_{n=0}^{\infty}\frac{1}{2\mathcal{E}z^r(t)^2}\left(1-\frac{2M}{z^r(t)}\right)^2\left(\frac{dz^r(t)/dt}{1-2M/z^r(t)}\right)^{2n}$$
$$= -\frac{q^2\mathcal{L}\mathcal{E}\partial_\alpha(r-z^r(t))}{4\pi z^r(t)(z^r(t)-2M)(1+\mathcal{L}^2/z^r(t)^2)}, \qquad \text{(B·9)}$$

where we have used the equation of motion, Eq. (3·14).

Appendix C
—— Absence of Large Powers of ℓ ——

We have $z=\omega r$ and $\epsilon=2M\omega$, which contain ω implicitly. This ω is replaced by a time differentiation, which produces $m\Omega_\varphi$. That is, ω is effectively of $O(\ell)$. In other words, we should regard z to be $O(\ell)$ and ϵ to be $O(\ell)$. Then, considering the radial function given in Eq. (2·8), one might think that the final expression for the force would have terms with large positive powers of ℓ. We explain here why this is actually **NOT** the case, by analyzing the leading ℓ behavior of Φ^ν and $\Phi^{-\nu-1}$.

Let us introduce Ψ^ν by

$$\Phi^\nu = \exp\left[\int^z dz\Psi^\nu\right] = \exp\left[\int^r dr\,\omega\,\Psi^\nu\right]. \qquad \text{(C·1)}$$

Then the equation for Ψ^ν gives

$$\Psi^\nu = -\frac{\nu}{z}+\left[\frac{\ell(\ell+1)}{z(z-\epsilon)}-\frac{z^2}{(z-\epsilon)^2}-\frac{\nu}{z(z-\epsilon)}-\frac{2z-\epsilon}{z(z-\epsilon)}\Psi^\nu-\partial_z\Psi^\nu\right]^{1/2}. \qquad \text{(C·2)}$$

Applying the large ℓ asymptotic expansion to this expression, we have

$$\Psi^\nu = \Psi_0^\nu+\delta\Psi^\nu\,; \quad \Psi_0^\nu = -\frac{\nu}{z}+\left[\frac{\ell^2}{z(z-\epsilon)}-\frac{z^2}{(z-\epsilon)^2}\right]^{1/2}, \qquad \text{(C·3)}$$

where $\Psi_0^\nu = O(\ell^0)$ and $\delta\Psi^\nu = O(1/\ell)$.

We find that, at the leading order in ℓ, the condition that the term of $O(1/z)$ in Ψ_0^ν should vanish determines ν in the sense of the PN expansion:

$$\nu = \nu_0 + O(\ell^0), \quad \nu_0 = \ell - \frac{15\epsilon^2}{16\ell}+\cdots. \qquad \text{(C·4)}$$

Also, Ψ_0^ν is given by

$$\Psi_0^\nu = -\frac{z}{2\ell}-\frac{z^3}{8\ell^3}-\frac{z^5}{16\ell^5}+O(z^7/\ell^7)$$
$$+\left(\frac{\ell^2}{2z^2}-\frac{3}{4}-\frac{5z^2}{16\ell^2}+O(z^4/\ell^4)\right)\frac{\epsilon}{\ell}+\left(\frac{3\ell^3}{8z^3}+O(z/\ell)\right)\frac{\epsilon^2}{\ell^2}+\cdots. \qquad \text{(C·5)}$$

Substituting this expression into Eq. (C·1), the result is

$$\Phi^\nu = 1 - \frac{z^2}{4\ell} - \frac{\ell\epsilon}{2z} + \frac{z^4}{32\ell^2} + \frac{z\epsilon}{8} + \frac{\ell^2\epsilon^2}{8z^2} + \cdots , \tag{C·6}$$

which coincides with Eq. (2·8) in the large ℓ limit. Another independent solution $\Psi_0^{-\nu-1}$ can be obtained through the replacement $\nu \to -\nu-1$, $\ell \to -\ell-1$, $+[\cdots]^{1/2} \to -[\cdots]^{1/2}$ in Eq. (C·3).

The product of $\Phi^\nu(r)$ and $\Phi^{-\nu-1}(r')$, which appears in the Green function, becomes

$$\Phi^\nu(z)\Phi^{-\nu-1}(z') = \exp\left(\int^z dz\, \Psi_0^\nu(z) + \int^{z'} dz'\, \Psi_0^{-\nu-1}(z')\right) \times O(\ell^0)\,. \tag{C·7}$$

As clearly seen from the expanded form of Ψ_0^ν given in Eq. (C·5), we have $\Psi_0^\nu(z) = (\ell/\omega)\sum_{n=0}^\infty (\omega/\ell)^{2n} C_n(r)$, where the quantities C_n are functions of r, and $\Psi_0^{-\nu-1}(z) = -\Psi_0^\nu(z) + O(1/\ell)$. This implies that, when z and z' are sufficiently close, we have

$$\int^z dz\, \Psi_0^\nu(z) + \int^{z'} dz'\, \Psi_0^{-\nu-1}(z') = \ell\,(r-r')\sum_{n=0}^\infty (\omega/\ell)^n F_n(r,r') + O(\ell^0), \tag{C·8}$$

where the F_n are functions of r and r' that are independent of ω and ℓ and regular in the coincidence limit, $r \to r'$. Therefore, we obtain

$$\Phi^\nu(z)\Phi^{-\nu-1}(z') = \sum_{s=1}^\infty \sum_{n=0}^\infty \ell^s (r-r')^s (\omega/\ell)^n \tilde{F}_{s,n}(r,r') + O(\ell^0), \tag{C·9}$$

where the $\tilde{F}_{s,n}$ are functions of r and r'. The terms apparently of $O(\ell^s)$ are always associated with the factor $(r-r')^s$. Therefore, the time differentiation must act on $r' = z^r(t')$ at least s times. Otherwise such terms vanish in the coincidence limit. Each time differentiation, however, reduces the power of ω by 1, and hence it produces the factor $1/\ell$. Thus, the terms that appear to be $O(\ell^s)$ turn out to be $O(\ell^0)$ in the end.

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